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SMARANDACHE FUNCTION

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SOME REMARKS ON THE SMARANDACHE FUNCTION

by

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1. **On the method of calculus proposed by Florentin Smarandache.** In [6] is defined a numerical function $S: N^* \to N$, as follows:
S(n) is the smallest nonnegative integer such that S(n)! is divisible by n.

For example $S(1) = 0$, $S(2^{12}) = 16$.

This function characterizes the prime numbers in the sense that $p > 4$ is prime if and only if $S(p) = p$. As it is showed in [6] this function may be extended to all integers by defining $S(-n) = S(n)$. If a and b are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$ More general, if $[a,b]$ is the last common multiple of a and b then

$$S([a,b]) = \max\{S(a), S(b)\} \tag{1}$$

So, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$ is the factorization of n into primes, then

$$S(n) = \max\{S(p_i^{\alpha_i}) \mid i = 1, \ldots, t\} \tag{2}$$

For the calculus of $S(p_i^{\alpha_i})$ in [6] it is used the fact that if $a = (p^n - 1)/(p - 1)$ then $S(p^a) = p^n$.
This equality results from the fact, if $\alpha_p(n)$ is the exponent of the prime p in the decomposition of n! into primes then

$$\alpha_p(n) = \sum_{i \geq 1} \left[\frac{n}{p^i} \right] \tag{3}$$

From (3) is results that $S(p^\alpha) \leq p \cdot \alpha$.

Now, if we note $a_n(p) = (p^n - 1)/(p - 1)$ then

$$S\left(p^{k_{m_1} a_{m_1}(p) + k_{m_2} a_{m_2}(p) + \ldots + k_{m_l} a_{m_l}(p)}\right) = k_{m_1} p^{m_1} + k_{m_2} p^{m_2} + \ldots + k_{m_l} p^{m_l} \tag{4}$$

for $k_{m_1}, k_{m_2}, \ldots, k_{m_{l-1}} \in \overline{1, p-1}$ and $k_{m_l} \in \{1, 2, \ldots, p\}$.

That is, if we consider the generalized scale

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

and the standard scale

$$(p) : 1 \,,\, p \,,\, p^2 \,,\, \ldots \,,\, p^n \,,\, \ldots$$

and we express the exponent a in the scale $[p]$, $a_{[p]} = \overline{k_{m_1} k_{m_2} \ldots k_{m_t}}$, then the left hand of the equality (4) is $S\left(p^{a_{[p]}}\right)$ and the right hand becomes $p\left(a_{[p]}\right)_{(p)}$. In other words, the right hand of (4) is the number obtained multiplying by p the exponent a writed in the scale $[p]$, readed it in the scale (p) .

So, (4) may be written as

$$S\left(p^{a_{[p]}}\right) = p\left(a_{[p]}\right)_{(p)} \qquad (5)$$

For example, to calculate $S(3^{89})$ we write the exponent a=89 in the scale

$$[3] : 1,\ 4\ ,\ 13\ ,\ 40\ ,\ 121\ ,\ \ldots$$

and so

$a_{m_1}(p) \le a \Leftrightarrow \left(p^{m_1} - 1\right)/\left(p-1\right) \le a \Leftrightarrow p^{m_1} \le \left(p-1\right)\cdot a + 1 \Leftrightarrow m_1 \le \log_p\left(\left(p-1\right)\cdot a + 1\right)$.

It results that m_1 is the integer part of $\log_p\left(\left(p-1\right)\cdot a + 1\right)$.

For our example $m_1 = \left[\log_3(2a+1)\right] = \log_3 179 = 4$. Then first digit of $a_{[3]}$ is $k_4 = \left[a/a_4(3)\right] = 2$. So, $89 = 2a_4(3) + 9$.

For $a_1 = 9$ it results $m_2 = \left[\log_3(2a_1+1)\right] = 2$, $k_2 = \left[a_1/a_2(3)\right] = 2$ and so $a_1 = 2a_2(3) + 1$.

Then $89 = 2a_4(3) + 2a_2(3) + a_1(3) = 2021_{[3]}$, and $S\left(3^{89}\right) = 3\left(2021\right)_{(3)} = 183$.

Indeed, $\sum_{i\ge1}\left[\frac{183}{3^i}\right] = 61 + 20 + 6 + 2 = 89$.

Let us observe that the calculus in the generalized scale $[p]$ is essentially different from the calculus in the standard scale (p) . That because if we note $b_n(p) = p^n$ then it results

$$a_{n+1}(p) = pa_n(p) + 1 \quad \text{and} \quad a_{n+1}\left(p\right) = pa_n\left(p\right) + 1 \qquad (6)$$

For this, to add some numbers in the scale $[p]$ we do as follows. We start to add from the digits of "decimals", that is from the column of $a_2(p)$. If adding some digits it is obtained $pa_2(p)$ then it is utilized a unit from the class of units (coefficients of $a_1(p)$) to obtain $pa_2(p) - 1 = a_3(p)$. Continuing to add, if agains it is obtained $pa_2(p)$, then a new unit must be used, from the class of units, etc.

For example if $m_{[5]} = 442$, $n_{[5]} = 412$ and $r_{[5]} = 44$ then

$$\begin{array}{r} m+n+r = 442+ \\ 412 \\ \underline{44} \\ \text{dcba} \end{array}$$

We start to add from the column corresponding to $a_2(5)$:

$$4a_2(5)+a_2(5)+4a_2(5) = 5a_2(5)+4a_2(5) .$$

Now utilizing a unit from the first column we obtain

$$5a_2(5)+4a_2(5) = a_3(5)+4a_2(5) \text{ , so } b = 4 .$$

Continuing, $4a_3(5)+4a_3(5)+a_3(5) = 5a_3(5)+4a_3(5)$ and using a new unit it results $4a_3(5)+4a_3(5)+a_3(5) = a_4(5)+4a_3(5)$, so $c = 4$ and $d = 1$. Finally, adding the remained units $4a_1(5)+2a_1(5) = 5a_1(5)+a_1(5) = 5a_1(5)+1 = a_2(5)$ it results that b must be modified and $a = 0$. So $m+n+r = 1450_5$.

We have applied the formula (5) to the calculus of the values of S for any integer between $N_1 = 31,000,000$ and $N_2 = 31,001,000$. A program has been designed to generate the factorization of every integer $n \in [N_1, N_2]$ (TIME (minutes) : START : 40:8:93, STOP : 56:38:85, more than 16 minutes) .

Afterwards, the Smarandache function has been calculated for every $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_t^{a_t}$ as follows :

1) max $p_i \cdot a_i$ is determined
2) $S_0 = S(p_i^{a_i})$, for i determined above
3) Because $S(p_j^{a_j}) \le p_j \cdot a_j$, we ignore the factors for which $p_j \cdot a_j \le S_0$.
4) Are calculated $S(p_j^{a_j})$ for $p_j \cdot a_j > S_0$ and is determined the greatest of these values.

(TIME (minutes): START: 25:52:75, STOP: 25:55:27, leas than 3 seconds)

2. Some diofantine equations concerning the function S.

In this section we shall apply the formula (5) for the study of the solutions of some diofantine equations proposed in (6).

a) Using (5) it can be proved that the diofantine equation

$$S(x \cdot y) = S(x) + S(y) \qquad (7)$$

has infinitely many solutions. Indeed, let us observe that from (2) every relatively prime integers x_0 and y_0 can' t be a solution from (7). Let now $x = p^a \cdot A$, $y = p^b \cdot B$ be such that $S(x) = S(p^a)$ and $S(y) = S(p^b)$.

Then $S(x \cdot y) = S(p^{a+b})$ and (7) becomes

$$p\left((a+b)_{[p]}\right)_{(p)} = p\left(a_{[p]}\right)_{(p)} + p\left(b_{[p]}\right)_{(p)}$$

or

$$\left((a+b)_{[p]}\right)_{(p)} = \left(a_{[p]}\right)_{(p)} + \left(b_{[p]}\right)_{(p)} \qquad (8)$$

There exists infinitely many values for a and b satisfying yhis equality. For example $a = a_3(p) = 100_{[p]}$, $b = a_2(p) = 10_{[p]}$ and (8) becomes $\left(110_{[p]}\right)_{(p)} = \left(100_{[p]}\right)_{(p)} + \left(10_{[p]}\right)_{(p)}$.

b) We shall prove now that the equation

$$S(x \cdot y) = S(x) \cdot S(y)$$

has no solution $x, y > 1$.

Let $m = S(x)$ and $n = S(y)$. It is sufficient to prove that $S(x \cdot y) = m \cdot n$. But it is said that $m! \cdot n!$ divide $(m+n)!$, so

$$(m \cdot n)! \;\vdots\; (m+n)! \;\vdots\; m! \cdot n! \;\vdots\; x \cdot y$$

c) If we note by (x, y) the greatest common divisor of x and y, then the ecuation

$$(x, y) = (S(x), S(y)) \qquad \mathbf{(9)}$$

has infinitelly many solutions. Indeed, because $x \geq S(x)$, the equality holding if and only if x is a prime it results that (9) has as solution every pair x, y of prime numbers and also every pair of product of prime numbers.

Let now $S(x) = p\left(a_{[p]}\right)_{(p)}$, $S(y) = q\left(b_{[q]}\right)_{(q)}$ be such that $(x, y) = d > 1$. Then because $(p.q) = 1$, if

$$a_1 = \left(a_{[p]}\right)_{[p]} \quad , \; b_1 = \left(b_{[p]}\right)_{[p]} \quad \text{and} \quad (p, b_1) = (a_1, q) = 1.$$

it result that the equality (9) becomes

$$\left(\left(a_{[p]}\right)_{(p)}, \left(b_{[q]}\right)_{(q)}\right) = d$$

and it is satisfied for various positive integers a and b. For example if $x = 2 \cdot 3^a$ and $y = 2 \cdot 5^b$ it results $d = 2$ and the equality $\left(\left(a_{[3]}\right)_{(3)}, \left(b_{[5]}\right)_{(5)}\right) = 2$ is satisfied for many values of $a, b \in N$.

d) If $[x, y]$ is the least common multiple of x and y then the equation

$$[x, y] = [S(x), S(y)] \qquad (10)$$

has as solutions every pair of prime numbers. Now, if x and y are composite numbers such that $S(x) = S\left(p_i^{a_i}\right)$ and $S(y) = S\left(p_j^{a_j}\right)$ with $p_i \neq p_j$ then the pair x, y can't be solution of the equation because in this case we have

$$[x, y] > p_i^{a_i} \cdot p_j^{a_j} > S(x) \cdot S(y) \geq S(x), S(y)$$

and if $x = p^a \cdot A$ and $y = p^b \cdot B$ with $S(x) = S\left(p^a\right)$, $S(y) = S\left(p^b\right)$ then

$$[S(x), S(y)] = \left[p\left(a_{[p]}\right)_{(p)}, p\left(b_{[p]}\right)_{(p)}\right] = p \cdot \left(a_{[p]}\right)_{(p)}, \left(b_{[p]}\right)_{(p)}$$

and $[x,y] = p^{\min(a,b)} \cdot [A,B]$ so (10) is satified also for many values of non relatively prime integers.

e) Finaly we consider the equation

$$S(x) + y = x + S(y)$$

which has as solution every pair of prime numbers, but also the composit numbers $x = y$. It can be found other composit number as solutions. For example if p and q are consecutive prime numbers such that

$$q - p = h > 0 \tag{11}$$

and $x = p \cdot A$, $y = q \cdot B$ then our equatic is equivalent to

$$y - x = S(y) - S(x) \tag{12}$$

If we consider the diofantine equation $qB - pA = h$ it results from (11) that $A_0 = B_0 = 1$ is a particular solution, so the general solution is $A = 1 + rq$, $B = 1 + rp$,for arbitrary integer r. Then for $r = 1$ it results $x = p(1+q)$, $y = q(1+p)$ and $y - x = h$. In addition, because p and q are consecutive primes it results that $p+1$ and $q+1$ are composite and so

$$S(x) = p \;,\; S(y) = q \;,\; S(y) - S(x) = h$$

and (12) holds.

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SMARANDACHE NUMERICAL FUNCTIONS

by

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F. Smarandache defines [1] a numerical function $S : \mathbb{N}^* \longrightarrow \mathbb{N}$. $S(n)$ is the smallest integer m such that $m!$ is divisible by n. Using certain results on standardised structures, three kinds of Smarandache functions are defined and are etablished some compatibility relations between these functions.

1. **Standardising functions.** Let X be a nonvoid set, $r \subset X \times X$ an equivalence relation, $\hat{X}$ the corresponding quotient set and $(I, \leq)$ a totally ordered set.

1.1 **Definition.** If $g : \hat{X} \longrightarrow I$ is an arbitrarely injective function, then $f : X \longrightarrow I$ defined by $f(x) = g(\hat{x})$ is a standardising function. In this case the set X is said to be $[r, (I, \leq), f]$ standardised.
If r_1 and r_2 are two equivalence relations on X, then $r = r_1 \wedge r_2$ is defined as $x\, r\, y$ if and only if $x\, r_1\, y$ and $x\, r_2\, y$. Of course r is an equivalence relation.

In the following theorem we consider functions having the same monotonicity. The functions $f_i : X \longrightarrow I$, $i = \overline{1,s}$ are of the same monotonicity if for every x, y from X it results

$$f_k(x) \leq f_k(y) \quad \text{if and only if} \quad f_j(x) \leq f_j(y) \quad \text{for} \quad k, j = \overline{1,s}$$

1.2 Theorem. If the standardising functions $f_i : X \longrightarrow I$ corresponding to the equivalence relations r_i, $i = \overline{1,s}$, are of the some monotonicity then $f = \max_i \{ f_i \}$ is a standardising function corresponding to $r = \bigwedge_i r_i$, having the same monotonicity as f_i.

Proof. We give the proof of theorem in case $s = 2$. Let $\hat{x}_{r_1}$, $\hat{x}_{r_2}$, $\hat{x}_r$ be the equivalence clases of x corresponding to r_1, r_2 and to $r = r_1 \wedge r_2$ respectively and $\hat{X}_{r_1}$, $\hat{X}_{r_2}$, $\hat{X}_r$ the quotient sets on X.

We have $f_1(x) = g_1(\hat{x}_{r_1})$ and $f_2(x) = g_2(\hat{x}_{r_2})$, where

$g_i : \hat{X}_{r_i} \longrightarrow I$, $i=1,2$ are injective functions. The function $g : \hat{X}_r \longrightarrow I$ defined by $g(\hat{x}_r) = \max\{g_1(\hat{x}_{r_1}), g_2(\hat{x}_{r_2})\}$ is injective.

Indeed, if $\hat{x}^1_r \neq \hat{x}^2_r$ and $\max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} =$
$= \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\}$, then be cause of the injectivity of g_1 and g_2 we have for example $\max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} =$
$= g_1(\hat{x}^1_{r_1}) = g_2(\hat{x}^2_{r_2}) = \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\}$ and we obtain a

contradiction because $f_1(x^2) = g_1(\hat{x}^2_{r_1}) < g_1(\hat{x}^1_{r_1}) = f_1(x^1)$

$$f_2(x^1) = g_2(\hat{x}^1_{r_1}) < g_2(\hat{x}^2_{r_2}) = f_2(x^2)$$, that is

f_1 and f_2 are not of the same monotonicity From the injectivity of g it results that $f : X \longrightarrow I$ defined by $f(x) = g(\hat{x}_r)$ is a standardising function. In addition we have $f(x^1) \leq f(x^2) \iff$
$g(\hat{x}^1_r) \leq g(\hat{x}^2_r) \iff \max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} \leq \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\} \iff$
$\iff \max\{f_1(x^1), f_2(x^1)\} \leq \max\{f_1(x^2), f_2(x^2)\} \iff f_1(x^1) \leq f_1(x^2)$ and $f_2(x^1) \leq f_2(x^2)$ because f_1 and f_2 are of the same monotonicity.

Let us supose now that $\top$ and $\perp$ are two algebraic lows on X and I respectively.

1.3. Definition. The standardising function $f: X \longrightarrow I$ is said to be Σ-compatibile with $\top$ and $\perp$ if for every x, y in X the triplet $(f(x), f(y), f(x \top y))$ satisfies the condition Σ. In this case it is said that the function f Σ-standardise the structrure $(X, \top)$ in the structure $(I, \leq, \perp)$.

For example, if f is the Smarandache function $S: \mathbb{N}^* \longrightarrow \mathbb{N}$, ($S(n)$ is the smallest integer such that $(S(n))!$ is divisible by n) then we get the following Σ-stadardisations:

a) S Σ_1-standardise $(\mathbb{N}^*, .)$ in $(\mathbb{N}^*, \leq, +)$ because we have

$$\Sigma_1: S(a.b) \leq S(a) + S(b)$$

b) but S verifie also the relation

$$\Sigma_2; \ \max(S(a), S(b)) \leq S(a.b) \leq S(a).S(b)$$

so S Σ_2-standardise the structure $(\mathbb{N}^*, .)$ in $(\mathbb{N}^*, \leq, .)$

2. Smarandache functions of first kind. The Smarandache function S is defined by means of the following functions S_p; for every prime number p let $S_p: \mathbb{N}^* \longrightarrow \mathbb{N}^*$ having the property that $(S_p(n))!$ is divisible by p^n and is the smallest positive integer with this property. Using the notion of standardising functions in this section we give some generalisasion of S_p.

2.1. Definition. For every $n \in \mathbb{N}^*$ the relation $r_n \subset \mathbb{N}^* \times \mathbb{N}^*$ is defined as follows: i) if $n = u^i$ ($u=1$ or $u=p$ number prime, $i \in \mathbb{N}^*$) and $a, b \in \mathbb{N}^*$ then $a \, r_n \, b$ if and only if it exists $k \in \mathbb{N}^*$ such that $k! = M u^{ia}$, $k! = M u^{ib}$ and k is the smallest positive integer with this property.

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_s^{i_s}$, then

$$r_n = r_{p_1^{i_1}} \wedge r_{p_2^{i_2}} \wedge \ldots \wedge r_{p_s^{i_s}}$$

2.2. Definition. For each $n \in \mathbb{N}^*$ the Smarandache function of first kind is the numerical function $S_n : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined as follows

i) if $n = u^i$ ($u=1$ or $u=p$ number prime) then $S_n(a) = k$, k being the smallest positive integer with the property that $k! = M\, u^{ia}$

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_s^{i_s}$, then $S_n(a) = \max\limits_{1 \le j \le s} \{ S_{p_j^{i_j}}(a) \}$

Let us observe that :

a) the functions S_n are standardising functions corresponding to the equivalence relations r_n and for $n=1$ we get $\hat{x}_{r_1} = \mathbb{N}^*$ for every $x \in \mathbb{N}^*$ and $S_1(n) = 1$ for every n.

b) if $n=p$ then S_n is the function S_p defined by Smarandache.

c) the functions S_n are increasing and so, are of the same monotonicity in the sense given in the above section.

2.3. Theorem. The functions S_n, for $n \in \mathbb{N}^*$, Σ_1-standardise $(\mathbb{N}^*, +)$ in $(\mathbb{N}^*, \le, +)$ by Σ_1: $\max(S_n(a), S_n(b)) \le S_n(a+b) \le S_n(a) + S_n(b)$ for every $a, b \in \mathbb{N}^*$ and Σ_2-standardise $(\mathbb{N}^*, +)$ in $(\mathbb{N}^*, \le, \cdot)$ by Σ_2 : $\max(S_n(a), S_n(b)) \le S_n(a+b) \le S_n(a) \cdot S_n(b)$, for every $a, b \in \mathbb{N}^*$

Proof. Let, for instance, p be a prime number, $n=p^i$, $i \in \mathbb{N}^*$ and $a^* = S_{p^i}(a)$, $b^* = S_{p^i}(b)$, $k = S_{p^i}(a+b)$. Then by the definition of S_n (Definition 2.2.) the numbers a^*, b^*, k are the smallest positive integers such that $a^*! = Mp^{ia}$, $b^*! = Mp^{ib}$ and $k! = Mp^{i(a+b)}$.

Because $k! = Mp^{ia} = Mp^{ib}$ we get $a^* \le k$ and $b^* \le k$, so $\max(a^*, b^*) \le k$

That is the first inequalities in Σ_1 and Σ_2 holds.

Now, $(a^*+b^*)! = a^*!(a^*+1) \cdot \ldots \cdot (a^* + b^*) = Ma^*!\, b^*! = Mp^{i(a+b)}$ and

so $k \le a^* + b^*$ which implies that Σ_1 is valide.

If $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_s^{i_s}$, from the first case we have

Σ_1: $\max\{S_{p_j^{i_j}}(a), S_{p_j^{i_j}}(b)\} \le S_{p_j^{i_j}}(a+b) \le S_{p_j^{i_j}}(a) + S_{p_j^{i_j}}(b), j=\overline{1,s}$

in consequence

$\max\{\max_j S_{p_j^{i_j}}(a), \max_j S_{p_j^{i_j}}(b)\} \le \max_j\{S_{p_j^{i_j}}(a+b)\} \le \max_j\{S_{p_j^{i_j}}(a)\} +$

$\max_j\{S_{p_j^{i_j}}(b)\}$, $j = \overline{1,s}$. That is

$$\max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

For the proof of the second part in Σ_2 let us notice that

$(a+b)! \le (ab)! \iff a+b \le ab \iff a > 1$ and $b > 1$ and that

ours inequality is satisfied for $n=1$ because $S_1(a+b)=S_1(a)=$
$= S_1(b) = 1$.

Let now $n>1$. It results that for $a^* = S_n(a)$ we have $a^* > 1$. Indeed, if $n = p_1^{i_1} p_2^{i_2} \cdots p_s^{i_s}$ then $a^* = 1$ if and only if $S_n(a)=$
$= \max_j \{S_{p_j^{i_j}}(a)\} = 1$ which implies that $p_1=p_2= \cdots =p_s= 1$,

so $n=1$. It results that for every $n>1$ we have $S_n(a)= a^*>1$ and

$S_n(b)= b^* > 1$. Then $(a^*+b^*)! \le (a^*.b^*)!$ we obtain

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a).S_n(b) \text{ from } n > 1.$$

3. Smarandache functions of the second kind. For every $n \in \mathbb{N}^*$, let S_n by the Smarandache function of the first kind defined above.

3.1. Definition. The Smarandache functions of the second kind are the functions $S^k : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined by $S^k(n)=S_n(k)$, for $k \in \mathbb{N}^*$.

We observe that for $k=1$ the function S^k is the Smarandache function S defined in [1], with the modify $S(1)= 1$. Indeed for.

$n>1$ $\quad S^1(n) = S_n(1) = \max_j\{S_{p_j^{i_j}}(1)\} = \max_j\{S_{p_j}(i_j)\} = S(n)$.

3.2. Theorem. The Smarandache functions of the second kind Σ_3-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\le,+)$ by

Σ_3: $\max\{S^k(a),S^k(b)\} \le S^k(a.b) \le S^k(a) + S^k(b)$, for every $a,b \in \mathbb{N}^*$

and Σ_4-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\le,\cdot)$ by

Σ_4: $\max\{S^k(a),S^k(b)\} \le S^k(a.b) \le S^k(a).S^k(b)$, for every $a,b\in\mathbb{N}^*$

Proof. The equivalence relation corresponding to S^k is r^k, defined by $a\,r^k b$ if and only if there exists $a^* \in \mathbb{N}^*$ such that $a^*! = Ma^k$, $a^*! = Mb^k$ and a^* is the smallest integer with this property.

That is, the functions S^k are standardising functions attached to the equivalence relations r^k.

This functions are not of the some monotonicity because, for example, $S^2(a) \le S^2(b) \leftrightarrow S(a^2) \le S(b^2)$ and from these inequalities $S^1(a) \le S^1(b)$ does not result.

Now for every $a,b \in \mathbb{N}^*$ let $S^k(a) = a^*$, $S^k(b) = b^*$, $S^k(a.b) = s$.

Then a^*, b^*, s are respectively these smallest positive integers such that $a^*! = Ma^k$, $b^*! = Mb^*$, $s! = M(a^k b^k)$ and so $s! = Ma^k = Mb^k$, that is, $a^* \le s$ and $b^* \le s$, which implies that $\max\{a^*,b^*\} \le s$

$$\text{or} \qquad \max\{S^k(a),S^k(b)\} \le S^k(a.b) \tag{3.1}$$

Because of the fact that $(a^*+b^*)! = M(a^*!\,b^*!) = M(a^k b^k)$, it results that $s \le a^* + b^*$, so

$$S^k(a.b) \le S^k(a) + S^k(b) \tag{3.2}$$

From (3.1) and (3.2) it results that

$$\max\{S^k(a),S^k(b)\} \le S^k(a) + S^k(b) \tag{3.3}$$

Which is the relation Σ_3.

From $(a^* b^*)! = M(a^*!.b^*!)$ it results that $S^k(a.b) \le S^k(a).S^k(b)$ and thus the relation Σ_4.

4. The Smarandache functions of the third kind.

We considere two arbitrary sequances (a) $1=a_1, a_2, \ldots, a_n, \ldots$

(b) $1=b_1, b_2, \ldots, b_n, \ldots$

with the properties that $a_{kn} = a_k . a_n$, $b_{kn} = b_k . b_n$. Obviously, there are infinitely many such sequences; because chosing an arbitrary value for a_2, the next terms in the net can be easily determined by the imposed condition.

Let now the function $f_a^b : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined by $f_a^b(n) = S_{a_n}(b_n)$, S_{a_n} is the Smarandache function of the first kind. Then it is easily to see that :

(i) for $a_n = 1$ and $b_n = n, n \in \mathbb{N}^*$ it results that $f_a^b = S_1$

(ii) for $a_n = n$ and $b_n = 1, n \in \mathbb{N}^*$ it results that $f_a^b = S^1$

4.1. Definition. The Smarandache functions of the third kind are the functions $S_a^b = f_a^b$ in the case that the sequances (a) and (b) are different from those concerned in the situation (i) and (ii) from above.

4.2. Theorem. The functions f_a^b Σ_3-standardise $(\mathbb{N}^*, \cdot)$ in $(\mathbb{N}^*, \leq, +, \cdot)$ by

$$\Sigma_3: \quad \max\{f_a^b(k), f_a^b(n)\} \leq f_a^b(k.n) \leq b_n . f_a^b(k) + b_k f_a^b(n) \quad .$$

Proof. Let $f_a^b(k) = S_{a_k}(b_k) = k^*$, $f_a^b(n) = S_{a_n}(b_n) = n^*$ and $f_a^b(kn) = S_{a_{kn}}(b_{kn}) = t$. Then k^*, n^* and t are the smallest positive integers such that $k^*! = M\, a_k^{b_k}$, $n^*! = M\, a_n^{b_n}$ and $t! = M\, a_{kn}^{b_{kn}} = M(a_k . a_n)^{b_k b_n}$. Of course,

$$\max\{k^*, n^*\} \leq t \tag{4.1}$$

Now, because $(b_k . n^*)! = M(n^*!)^{b_k}$, $(b_n . k^*)! = M(k^*!)^{b_n}$ and

$(b_k n^* + b_n k^*)! = M[(b_k n^*)!.(b_n k^*)!] = M[(n^*!)^{b_k}.(k^*!)^{b_n}] =$

$= M[(a_n^{b_n})^{b_k} . (a_k^{b_k})^{b_n}] = M[(a_k . a_n)^{b_k b_n}]$ it results that

$$t \le b_n k^* + b_k n^* \qquad (4.2)$$

From (4.1) and (4.2) we obtain

$$\max\{k^*, n^*\} \le t \le b_n k^* + b_k n^* \qquad (4.3)$$

From (4.3) we get Σ_3, so the Smarandache functions of the third kind satisfy

Σ_σ: $\max\{S_a^b(k), S_a^b(n)\} \le S_a^b(kn) \le b_n S_a^b(k) + b_k S_a^b(n)$, for evry $k,n \in \mathbb{N}^*$

4.3. Example. Let the sequances (a) and (b) defined by $a_n = b_n = n$, $n \in \mathbb{N}^*$.

The corresponding Smarandache function of the third kind is

$S_\alpha^a : \mathbb{N}^* \longrightarrow \mathbb{N}^*$, $S_a^a(n) = S_n(n)$ and Σ_σ becomes

$$\max\{S_k(k), S_n(n)\} \le S_{kn}(kn) \le nS_k(k) + kS_n(n) \text{, for every } k,n \in \mathbb{N}^*$$

This relation is equivalent with the following relation written by meens with the Smarandache function:

$$\max\{S(k^k), S(n^n)\} \le S[(kn)^{kn}] \le n.S(k^k) + k.S(n^n) .$$

References

[1] F.Smarandache, A Function in the Number Theory, An.Univ. Timisoara, seria st. mat Vol. XVIII, fasc.1, pp.79-88.1980.

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THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_\eta(n) = n$ (Ω)

by Pål Grønås

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 47 in [1]). The question is: "Are there an infinity of nonprimes n such that $\sigma_\eta(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of (Ω). As the wording of Problem 29916 indicates, (Ω) is satisfied if n is a prime. This is not the case for $n = 1$ because $\sigma_\eta(1) = 0$.

Suppose $\prod_{i=1}^k p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes, $r_i \in \mathbf{N}$ and $p_1 r_1 \geq p_i r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_1^{s_1}) \leq p_1 s_1$ we see that $p_1^{r_1} = n = \sigma_\eta(n) = \sigma_\eta(p_1^{r_1}) = \sum_{s_1=0}^{r_1} \eta(p_1^{s_1}) \leq \sum_{s_1=0}^{r_1} p_1 s_1 = \frac{p_1 r_1 (r_1+1)}{2}$. Therefore $2 p_1^{r_1-1} \leq r_1(r_1+1)$ (Ω_1) for some $r_1 \geq 2$. For $p_1 \geq 5$ this inequality (Ω_1) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of (Ω_1) we can find a supremum for r_1 depending on the value of p_1. For $p_1 = 2$ the actual candidates for r_1 are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of (Ω) in this case, namely $n = 4$, 8, 9 and 16. Calculating $\sigma_\eta(n)$ for each of these 4 values, we get $\sigma_\eta(4) = 6$, $\sigma_\eta(8) = 10$, $\sigma_\eta(9) = 9$ and $\sigma_\eta(16) = 16$. Consequently the only solutions of (Ω) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_\eta(n)$$

Substituting n with it's prime factorization we get

$$\begin{aligned}
\prod_{i=1}^k p_i^{r_i} &= \sigma_\eta\Big(\prod_{i=1}^k p_i^{r_i}\Big) = \sum_{\substack{d|n \\ d>0}} \eta(d) = \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \eta\Big(\prod_{i=1}^k p_i^{s_i}\Big) \\
&= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\,\eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k})\,\} \\
&\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\,p_1 s_1, \ldots, p_k s_k\,\} \quad \text{since} \quad \eta(p_i^{s_i}) \leq p_i s_i \\
&< \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\,p_1 r_1, \ldots, p_k r_k\,\} \quad \text{because} \quad s_i \leq r_i \\
&= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} p_1 r_1 \quad (p_1 r_1 \geq p_i r_i \text{ for } i \geq 2) \\
&\leq p_1 r_1 \prod_{i=1}^k (r_i + 1),
\end{aligned}$$

which is equivalent to

$$\prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{p_1 r_1(r_1+1)}{p_1^{r_1}} = \frac{r_1(r_1+1)}{p_1^{r_1-1}} \quad (\Omega_2)$$

This inequality motivates a closer study of the functions $f(x) = \frac{a^x}{x+1}$ and $g(x) = \frac{x(x+1)}{b^x-1}$ for $x \in [1, \infty)$, where a and b are real constants ≥ 2. The derivatives of these two functions are $f'(x) = \frac{a^x}{(x+1)^2}[(x+1)\ln a - 1]$ and $g'(x) = \frac{(-\ln b)x^2+(2-\ln b)x+1}{b^x-1}$. Hence $f'(x) > 0$ for $x \geq 1$ since $(x+1)\ln a - 1 \geq (1+1)\ln 2 - 1 = 2\ln 2 - 1 > 0$. So f is increasing on $[1, \infty)$. Moreover $g(x)$ reaches its absolute maximum value for $x = \max\{1, \frac{2-\ln b+\sqrt{(\ln b)^2+4}}{2\ln b} = \hat{x}\}$. Now $\sqrt{(\ln b)^2+4} < \ln b + 2$ for $b \geq 2$, which implies that $\hat{x} < \frac{(2-\ln b)+(\ln b+2)}{2\ln b} = \frac{2}{\ln b} \leq \frac{2}{\ln 2} < 3$. Futhermore it is worth mentioning that $f(x) \to \infty$ and $g(x) \to 0$ as $x \to \infty$.

Applying this to our situation means that $\frac{p_i^{r_i}}{r_i+1}$ $(i \geq 2)$ is strictly increasing from $\frac{p_i}{2}$ to ∞. Besides $\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}, \frac{12}{p_1^2}\} = \max\{2, \frac{6}{p_1}\} \leq 3$ because $\frac{6}{p_1} \geq \frac{12}{p_1^2}$ whenever $p_1 \geq 2$. Combining this knowledge with (Ω_2) we get that $\prod_{i=2}^{k} \frac{p_i}{2} \leq \prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1}-1} \leq 3$ (Ω_3) for all $r_1 \in \mathbb{N}$. In other words, $\prod_{i=2}^{k} \frac{p_i}{2} < 3$. Now $\prod_{i=2}^{4} \frac{p_i}{2} \geq \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4} > 3$, which implies that $k \leq 3$.

Let us assume $k = 2$. Then (Ω_2) and (Ω_3) state that $\frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}}$ and $\frac{p_2}{2} < 3$, i.e. $p_2 < 6$. Next we suppose $r_2 \geq 3$. It is obvious that $p_1 p_2 \geq 2 \cdot 3 = 6$, which is equivalent to $p_2 \geq \frac{6}{p_1}$. Using this fact we get $\frac{p_2^3}{4} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}\} \leq \max\{2, p_2\} = p_2$, so $p_2^2 < 4$. Accordingly $p_2 < 2$, a contradiction which implies that $r_2 \leq 2$. Hence $p_2 \in \{2, 3, 5\}$ and $r_2 \in \{1, 2\}$.

Futhermore $1 \leq \frac{p_2}{2} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1-1}}$, which implies that $r_1 \leq 6$. Consequently, by fixing the values of p_2 and r_2, the inequalities $\frac{r_1(r_1+1)}{p_1^{r_1-1}} > \frac{p_2^{r_2}}{r_2+1}$ and $p_1 r_1 \geq p_2 r_2$ give us enough information to determine a supremum (less than 7) for r_1 for each value of p_1.

This is just what we have done, and the result is as follows:

p_2	r_2	p_1	r_1	$n = p_1^{r_1} p_2^{r_2}$	$\sigma_\eta(n)$	IF $\sigma_\eta(n) = n$ THEN
2	1	3	$1 \leq r_1 \leq 3$	$2 \cdot 3^{r_1}$	$2 + 3r_1(r_1+1)$	$3 \mid 2$
2	1	5	$1 \leq r_1 \leq 2$	$2 \cdot 5^{r_1}$	$2 + 5r_1(r_1+1)$	$5 \mid 2$
2	1	$p_1 \geq 7$	1	$2p_1$	$2 + 2p_1$	$0 = 2$
2	2	3	2	36	34	$34 = 36$
2	2	$p_1 \geq 5$	1	$4p_1$	$3p_1 + 6$	$p_1 = 6$
3	1	2	$2 \leq r_1 \leq 5$	$3 \cdot 2^{r_1}$	$2r_1^2 - 2r_1 + 12$	$r_1 = 3$
3	1	$p_1 \geq 5$	1	$3p_1$	$2p_1 + 3$	$p_1 = 3$
5	1	2	3	40	30	$30 = 40$

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where $n = 3 \cdot 2^{r_1}$ and $r_1 = 3$. So $n = 3 \cdot 2^3 = 24$ and $\sigma_\eta(24) = 24$. In other words, $n = 24$ is the only solution of (Ω) when $k = 2$.

Finally, suppose $k = 3$. Then we know that $\frac{p_2}{2} \cdot \frac{p_3}{2} < 3$, i.e. $p_2 p_3 < 12$. Hence $p_2 = 2$ and $p_3 \geq 3$. Therefore $\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{3^{r_1-1}} \leq 2$ (Ω_4) and by applying (Ω_3) we find that $\prod_{i=2}^{3} \frac{p_i}{2} = \frac{p_3}{2} < 2$, giving $p_3 = 3$.

Combining the two inequalities (Ω_2) and (Ω_4) we get that $\frac{2^{r_2}}{r_2+1} \cdot \frac{3^{r_3}}{r_3+1} < 2$. Knowing that the left side of this inequality is a product of two strictly increasing functions on $[1, \infty)$, we see that the only possible choices for r_2 and r_3 are $r_2 = r_3 = 1$. Inserting these values in (Ω_2), we get $\frac{2^1}{1+1} \cdot \frac{3^1}{1+1} = \frac{3}{2} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{5^{r_1-1}}$. This implies that $r_1 = 1$. Accordingly (Ω) is satisfied only if $n = 2 \cdot 3 \cdot p_1 = 6\,p_1$:

$$\begin{aligned}
6\,p_1 &= \sigma_\eta(6\,p_1) \\
&= \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1}\sum_{j=0}^{1} \eta(2^i 3^j p_1) \\
&= 0 + 2 + 3 + 3 + \sum_{i=0}^{1}\sum_{j=0}^{1} \max\{\, \eta(p_1), \eta(2^i 3^j) \,\} \\
&= 8 + \sum_{i=0}^{1}\sum_{j=0}^{1} \max\{\, p_1, \eta(2^i 3^j) \,\} \\
&= 8 + 4\,p_1 \quad \text{because } \eta(2^i 3^j) \leq 3 < p_1 \ \text{ for all } \ i, j \in \{0, 1\} \\
&\Downarrow \\
p_1 &= 4
\end{aligned}$$

which contradicts the fact that $p_1 \geq 5$. Therefore (Ω) has no solution for $k = 3$.

Conclusion: $\sigma_\eta(n) = n$ if and only if n is a prime, $n = 9$, $n = 16$ or $n = 24$.

REMARK: A consequence of this work is the solution of the inequality $\sigma_\eta(n) > n$ $(*)$. This solution is based on the fact that $(*)$ implies (Ω_2).

So $\sigma_\eta(n) > n$ if and only if $n = 8, 12, 18, 20$ or $n = 2p$ where p is a prime. Hence $\sigma_\eta(n) \leq n + 4$ for all $n \in \mathbf{N}$.

Moreover, since we have solved the inequality $\sigma_\eta(n) \geq n$, we also have the solution of $\sigma_\eta(n) < n$.

References

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ON THE SUMATORY FUNCTION ASSOCIATED TO THE SMARANDACHE FUNCTION

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It is said that for every numerical function f it can be attashed the sumatory function :

$$F(n) = \sum_{d \mid n} f(d) \qquad (1)$$

The function f is expressed as :

$$f(n) = \sum_{uv=n} \mu(u) . F_f(v) \qquad (2)$$

Where μ is the Möbius function ($\mu(1)=1$, $\mu(n)=0$ if n is divisible by the square of a prime number , $\mu(n)=(-1)^k$ if n is the product of k different prime numbers)

If f is the Smarandache function and $n = p^\alpha$ then :

$$F_S(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j)$$

In [2] it is proved that

$$S(p^j) = (p - 1) \cdot j + \alpha_{[p]}(j) \qquad (3)$$

Where $\alpha_{[p]}(j)$ is the sum of the digits of the integer j,written in the generalised scale

$$[p] = a_1(p) \ , \ a_2(p) \ , \ \ldots \ , \ a_k(p) \ , \ \ldots$$

with $a_i(p) = (p^n - 1)/(p - 1)$

So

$$F_s(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j) = (p - 1) \frac{\alpha(\alpha + 1)}{2} + \sum_{j=1}^{\alpha} \alpha_{[p]}(j) \qquad (4)$$

Using the expresion of α given by (3) it results

$$(\alpha + 1)(S(p^\alpha) - \alpha_{[p]}(\alpha)) = 2(F_S(p^\alpha) - \sum_{j=1}^{\alpha} \alpha_{[p]}(j))$$

In the following we give an algorithm to calculate the sum in the right hand of (4). For this, let $\alpha_{(p)} = \overline{k_s . k_{s-1} . \ \dots \ . k_1}$ the expresion of α in the scale [p] and $j_{(p)} = \overline{k_{s_j} . k_{s_j -1} . \ \dots \ . k_1}$. We shall say that k_{i_j} are the digits of order i, for j = 1,2, ... ,α.

To calculate the sum of all the digits of order i,let $\nu_i = \alpha - a_i(p) + 1$.

Now we consider two cases :

(i) if $k_i \neq 0$, let :

$z_i(\alpha) = (\overline{k_s k_{s-1} \dots k_{i+1}})_{u=\alpha(p)}$, the equality $u = a_i(p)$ denothig that for the number writen between parantheses,the classe of units is $a_i(p)$.

Then $z_i(\alpha)$ is the number of all zeros of order i for the integers $j \leq \alpha$ and $\alpha_i = \nu_i(\alpha) - z_i(\alpha)$ is the number of the non-null digits.

(ii) if $k_i = 0$, let β the greatest number,less then α,having a non-null digit of order i.Then β is of the form :

$\beta_{(p)} = \overline{k_s k_{s-1} \dots k_{i+2}(k_{i+1}-1)p00 \dots 0}$ and of course $s_i(\alpha) = s_i(\beta)$.It results that there exist $\alpha_i(\beta)$ non-null digits of order i.

Let A_i, B_i, r_i, ρ_i given by equalities :

$$\alpha_i = A_i((p - 1)a_i(p) + 1) + r_i = A_i(a_{i+1}(p) - a_i(p)) + r_i$$

$$r_i = B_i a_i(p) + \rho_i$$

Then

$$s_i(\alpha) = A_i a_i(p) \frac{p(p - 1)}{2} + A_i p + a_i(p) \frac{B_i(B_i + 1)}{2} + \rho_i(B_i + 1)$$

and

$$\sum_{j=1}^{\alpha} \sigma_{(p)}(j) = \sum_{i=1}^{\alpha} s_i(\alpha) = \frac{p(p - 1)}{2} \sum_{i\geq 1} A_i a_i(p) + p \sum_{i\geq 1} A_i +$$

$$\frac{1}{2} \sum_{i\geq 1} a_i(p) B_i(B_i + 1) + \sum_{i\geq 1} \rho_i(B_i + 1)$$

For example if α = 149 and p = 3 it results :

[3] 1 , 4 , 13 , 40 , 121 , ...

$\alpha_{(3)} = 10202$, $\nu_1(\alpha) = (1020)_{u=\alpha_1(3)} = 48$ $\alpha_1 = \nu_1(\alpha) - z_1(\alpha) = 101$

For $\beta_{(3)} = 10130 = 146$ it results $\nu_2(\beta) = 143$, $z_2(\beta) =$

$(101)_{u=\alpha_2(3)} = u_3 + u = 3u_2 + 1 + u = 3(3u + 1) + 1 + u = 44$,

$\alpha_2 = 99$, $\nu_3(\alpha) = 137$, $z_3(\alpha) = (10)_{u=\alpha_3(3)} = 40$. $\alpha_3 = 97$.

For $\beta_{(3)} = 3000 = 120$ it results $\nu_4(\beta) = 81$, $z_4(\beta) = 0$, $\alpha_4 = 108$.

$\nu_5(\alpha) = 29$, $z_5(\alpha) = 0$, $\alpha_5 = 29$, and

$$A_1 = \left[\frac{\alpha_1}{\alpha_2 - \alpha_1}\right] = 33 \ , \ r_1 = 2 \ , \ B_1 = \left[\frac{z}{\alpha_1}\right] \ , \ \rho_1 = 0 \ , \ s_1 = 201$$

Analogously $s_2 = 165$, $s_3 = 145$, $s_4 = 123$ and $s_5 = 129$, so

$$\sum_{i=1}^{140} \alpha_{(3)}(i) = 633 \ , \ F_s(3^{140}) = 22983 .$$

Now let us consider $n = p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k$, with $p_1 < p_2 < \ldots < p_k$ prime numbers. Of course, $S(n) = p_k$ and from

$$F_s(1) = S(1) = 0$$

$$F_s(p_1) = S(1) + S(p_1) = p_1$$

$$F_s(p_1 \cdot p_2) = p_1 + 2p_2 = F(p_1) + 2p_2$$

$$F_s(p_1 \cdot p_2 \cdot p_3) = p_1 + 2p_2 + 2^2 p_3 = F(p_1 \cdot p_2) + 2^2 p_3$$

it results :

$$F_s(p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k) = F(p_1 \cdot p_2 \cdot \ \ldots \ p_{k-1}) + 2^{k-1} p^k$$

That is :

$$F(p_1 \cdot p_2 \cdot \ \ldots \ \cdot p_k) = \sum_{i=1}^{k} 2^{i-1} p_i$$

The equality (2) becomes :

$$p_k = S(n) = \sum_{uv=n} \mu(n) \ F_s(v) =$$

$$= F(n) - \sum_i F\left(\frac{n}{p_i}\right) + \sum_{i,j} F\left(\frac{n}{p_i p_j}\right) + \ldots + \sum_{i=1}^{k} F(p_i)$$

and became $F(p_i) = p_i$, it results :

$$F\left(\frac{n}{p_i}\right) = F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1} \cdot p_{i+1} \cdot \ \dots \ \cdot p_k) = \sum_{j=1}^{i-1} 2^{j-1} p_j + \sum_{j=i+1}^{k} 2^{j-1} p_j =$$

$$= F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdot \ \dots \ \cdot p_k).$$

Analogously,

$$F\left(\frac{n}{p_i p_j}\right) = F(p_1 \cdot p_2 \cdot \ \dots \ \cdot p_{i-1}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdot \ \dots \ \cdot p_{j-1}) +$$

$$+ 2^{j-1} F(p_{j+1} \ \dots \ \cdot p_k)$$

Finaly, we point out as an open problem that, by the Shapiro's theorem, if it exist a numerical function $g : \mathbb{N} \longrightarrow \mathbb{R}$ such that

$$g(n) = \sum_{d|n} P(n) \ S\left(\frac{n}{d}\right)$$

were P is a totaly multiplicative function and $P(1) = 1$, then

$$S(n) = \sum_{d|n} \mu(d) \ P(d) \ g\left(\frac{n}{d}\right)$$

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A proof of the non-existence of "Samma".

by Pål Grønås

Introduction: If $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of the natural number $n \geq 2$, then it is easy to verify that

$$S(n) = S(\prod_{i=1}^{k} p_i^{r_i}) = \max\{\ S(p_i^{r_i})\ \}_{i=1}^{k}.$$

From this formula we see that it is essensial to determine $S(p^r)$, where p is a prime and r is a natural number.

Legendres formula states that

$$n! = \prod_{i=1}^{k} p_i^{\sum_{m=1}^{\infty} [n/p_i^m]}. \tag{1}$$

The definition of the Smarandache function tells us that $S(p^r)$ is the least natural number such that $p^r \mid (S(p^r))!$. Combining this definition with (1), it is obvious that $S(p^r)$ must satisfy the following two inequalities:

$$\sum_{k=1}^{\infty} \left[\frac{S(p^r)-1}{p^k}\right] < \ r \ \leq \sum_{k=1}^{\infty} \left[\frac{S(p^r)}{p^k}\right]. \tag{2}$$

This formula (2) gives us a lower and an upper bound for $S(p^r)$, namely

$$(p-1)r + 1 \ \leq \ S(p^r) \ \leq \ pr. \tag{3}$$

It also implies that p divides $S(p^r)$, which means that

$$S(p^r) = p(r-i) \ \text{ for a particular } \ 0 \ \leq \ i \ \leq \ [\tfrac{r-1}{p}].$$

"Samma": Let $T(n) = 1 - \log(S(n)) + \sum_{i=2}^{n} \frac{1}{S(i)}$ for $n \geq 2$. I intend to prove that $\lim_{n\to\infty} T(n) = \infty$, i.e. "Samma" does not exists.

First of all we define the sequence $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_n =$ the nth prime.

Next we consider the natural number p_m^n. Now (3) gives us that

$$
\begin{aligned}
S(p_i^k) &\leq p_i\,k \quad \forall\, i \in \{1,\ldots,m\} \text{ and } \forall\, k \in \{1,\ldots,n\} \\
&\Downarrow \\
\frac{1}{S(p_i^k)} &\geq \frac{1}{p_i\,k} \\
&\Downarrow \\
\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{1}{S(p_i^k)} &\geq \sum_{i=1}^{m}\sum_{k=1}^{n}\frac{1}{p_i\,k} = \Big(\sum_{i=1}^{m}\frac{1}{p_i}\Big)\cdot\Big(\sum_{k=1}^{n}\frac{1}{k}\Big) \\
&\Downarrow
\end{aligned}
$$

$$\sum_{k=2}^{p_m^n}\frac{1}{S(k)} \geq \Big(\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{n}\frac{1}{k}\Big) \tag{4}$$

since $S(k) > 0$ for all $k \geq 2$, $p_a^b \leq p_m^n$ whenever $a \leq m$ and $b \leq n$ and $p_a^b = p_c^d$ if and only if $a = c$ and $b = d$.

Futhermore $S(p_m^n) \leq p_m\,n$, which implies that $-\log S(p_m^n) \geq -\log(p_m\,n)$ because $\log x$ is a strictly increasing function in the intervall $[2,\infty)$. By adding this last inequality and (4), we get

$$
\begin{aligned}
T(p_m^n) &= 1-\log(S(p_m^n)) + \sum_{i=2}^{p_m^n}\frac{1}{S(i)} \geq 1-\log(p_m\,n) + \Big(\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{n}\frac{1}{k}\Big) \\
&\Downarrow \\
T(p_m^{p_m}) &\geq 1-\log(p_m^2) + \Big(\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m}\frac{1}{k}\Big) \qquad (n = p_m) \\
&\Downarrow \\
T(p_m^{p_m}) &\geq 1 + 2\,\Big(-\log p_m + \sum_{k=1}^{p_m}\frac{1}{k}\Big) + \Big(-2+\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m}\frac{1}{k}\Big) \\
&\Downarrow \\
\lim_{m\to\infty} T(p_m^{p_m}) &\geq 1 + 2\cdot\lim_{m\to\infty}\Big(-\log p_m + \sum_{k=1}^{p_m}\frac{1}{k}\Big) + \lim_{m\to\infty}\Big[\Big(-2+\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m}\frac{1}{k}\Big)\Big] \\
&= 1 + 2\cdot\lim_{p_m\to\infty}\Big(-\log p_m + \sum_{k=1}^{p_m}\frac{1}{k}\Big) + \lim_{m\to\infty}\Big[\Big(-2+\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\Big(\sum_{k=1}^{p_m}\frac{1}{k}\Big)\Big] \\
&= 1 + 2\gamma + \lim_{m\to\infty}\Big(-2+\sum_{k=1}^{m}\frac{1}{p_k}\Big)\cdot\lim_{p_m\to\infty}\Big(\sum_{k=1}^{p_m}\frac{1}{k}\Big) \qquad (\gamma = \text{Euler's constant}) \\
&= \infty
\end{aligned}
$$

since both $\sum_{k=1}^{t}\frac{1}{k}$ and $\sum_{k=1}^{t}\frac{1}{p_k}$ diverges as $t\to\infty$. In other words, $\lim_{n\to\infty} T(n) = \infty$. $\square$

CALCULATING THE SMARANDACHE FUNCTION FOR POWERS OF A PRIME

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Introduction

The Smarandache function is an integer function, S, of an integer variable, n. S is the smallest integer such that S! is divisible by n. If the prime factorisation of n is known

$$n = \prod m_i^{\,p_i}$$

where the p_i are primes then it has been shown that

$$S(n) = \mathrm{Max}\left(S\left(m_i^{\,p_i}\right)\right)$$

so a method of calculating S for prime powers will be useful in calculating S(n).

The inverse function

It is easier to start with the inverse problem. For a given

prime, p, and a given value of S, a multiple of p, what is the maximum power, m, of p which is a divisor of S! ? If we consider the case p=2 then all even numbers in the factorial contribute a factor of 2, all multiples of 4 contribute another, all multiples of 8 yet another and so on.

m = S DIV2 + (S DIV2)DIV2 + ((S DIV2)DIV2)DIV2 + ...

In the general case

m = S DIVp + (S DIVp)DIVp + ((S DIVp)DIVp)DIVp + ...

The series terminates by reaching a term equal to zero. The Pascal program at the end of this paper contains a function invSpp to calculate this function.

Using the inverse function

If we now look at the values of S for succesive powers of a prime, say p=3,

m	1	2	3	4	5	6	7	8	9	10	...
	*	*		*	*	*		*	*	*	
S(3^m)	3	6	9	9	12	15	18	18	21	24	...

where the asterisked values of m are those found by the inverse function, we can see that these latter determine the points after which S increases by p. In the Pascal program the procedure tabsmarpp fills an array with the values of S for successive powers of a prime.

The Pascal program

The program tests the procedure by accepting a prime input from the keyboard, calculating S for the first 1000 powers, reporting the time for this calculation and entering an endless loop of accepting a power value and reporting the corresponding S value as stored in the array.

The program was developed and tested with Acornsoft ISO-Pascal on a BBC Master. The function 'time' is an extension to standard Pascal which delivers the timelapse since last reset in centi-seconds. On a computer with a 65C12 processor running at 2 MHz the 1000 S values are calculated in about 11 seconds, the exact time is slightly larger for small values of the prime.

```
program TestabSpp(input,output);
var t,p,x: integer;
Smarpp:array[1..1000] of integer;

function invSpp(prime,smar:integer):integer;
var m,x:integer;
begin
m:=0;
x:=smar;
repeat
x:=x div prime;
m:=m+x;
until x<prime;
invSpp:=m;
end; {invSpp}
```

```
procedure tabsmarpp(prime,tabsize:integer);
var i,s,is:integer;
exit:boolean;
begin
exit:=false;
i:=1;
is:=1;
s:=prime;
repeat
repeat
Smarpp[i]:=s;
i:=i+1;
if i>tabsize then exit:=true;
until (i>is) or exit;
s:=s+prime;
is:=invSpp(prime,s);
until exit;
end; {tabsmarpp}

begin
read(p);
t:=time;
tabsmarpp(p,1000);
writeln((time-t)/100);
repeat
read(x);
writeln('Smarandache for ',p,' to power ',x,' is ',Smarpp[x]);
until false;
end. {testabspp}
```

CALCULATING THE SMARANDACHE FUNCTION WITHOUT FACTORISING

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Introduction

The usual way of calculating the Smarandache function S(n) is to factorise n, calculate S for each of the prime powers in the factorisation and use the equation

$$S(n) = \text{Max}\left(S\left(m_i^{\,p_i}\right)\right)$$

This paper presents an alternative algorithm for use when S is to be calculated for all integers up to n. The integers are synthesised by combining all the prime powers in the range up to n.

The Algorithm

The Pascal program at the end of this paper contains a procedure tabsmarand which fills a globally declared array, Smaran, with the values of S for the integers from 2 to the limit specified by a parameter. The calculation is carried out in four stages.

Powers of 2

The first stage calculates S for those powers of 2 that fall within the limit and stores them in the array Smaran at the subscript which corresponds to the value of that power of 2. At the end of this stage the array contains S for:-

2,4,8,16,32....

interspersed with zeros for all the other entries.

General case

The next stage uses succesive primes from 3 upwards. For each prime the S values of the relevant powers of the prime, and also the values of the prime powers are calculated, and stored in the arrays Smarpp and Prpwr, by the procedure tabsmarpp. This procedure is essentially the same as that in a previous paper except that:

a) the calculation stops when the last prime power exceeds the limit

and b) the prime powers are also calculated and stored.

Then for each non-zero entry in Smarand that entry is multiplied by successive powers of the prime and the S values calculated and stored in Smarand. Both of these loops terminate on reaching the limit value. Finally the S values for the prime powers are copied into Smarand. After the prime 3 the array contains:-

2,3,4,0,3,0,4,6,0,0,4....

This process is followed for each prime up to the square root of the limit. This general case could be continued up to the limit but it is more efficient to stop at the square root and treat the larger primes as seperate cases.

Largest primes

The largest primes, those greater than half the limit, contribute only themselves, S(prime)=prime, to the array of Smarandache values.

Multiples of prime only

The intermediate case between the last two is for primes larger than the square root but smaller than half the limit. In this case no powers of the prime are needed, only multiples of those entries already in Smarand by the prime itself. The prime is then copied into the array.

The Pascal program

The main program calls tabsmarand to calculate S values then enters a loop in which two integers are input from the keyboard which specify a range of values for which the contents of the array are displayed for checking.

The program was developed and tested with Acornsoft ISO-Pascal on a BBC Master computer. The function 'time' delivers the time lapse (in centiseconds) since last reset. On a computer with a 65C12 processor running at 2MHz the following timings were obtained:-

limit	seconds
1000	6.56
2000	12.87
3000	19.19
4000	25.64
5000	31.80

In this range the times appear almost linear. It would be useful to have this confirmed or disproved on a larger, faster computer.

```
program Testsmarand(input,output);
const limit=5000;
var count,st,fin:integer;
Smaran:array[1..5001] of integer;

procedure tabsmarand(limit:integer);
var count,t,i,s,is,pp,prime,pwcount,mcount,multiple: integer;
exit: boolean;
Prpwr:array[1..12] of integer;
Smarpp:array[1..12] of integer;

function max(x,y: integer):integer;
begin
if x>y then max:=x else max:=y;
end; {max}

function invSpp(prime,smar:integer):integer;
var n,x:integer;
begin
n:=0;
x:=smar;
repeat
x:=x div prime;
n:=n+x;
until x<prime;
invSpp:=n;
end; {invSpp}

procedure tabsmarpp(prime,limit:integer);
var i,s,is,pp:integer;
exit:boolean;
begin
exit:=false;
pp:=1;
i:=1;
is:=1;
s:=prime;
repeat
repeat
Smarpp[i]:=s;
pp:=pp*prime;
Prpwr[i]:=pp;
i:=i+1;
if pp>limit then exit:=true;
until (i>is) or exit;
s:=s+prime;
is:=invSpp(prime,s);
until exit;
end; {tabsmarpp}
```

```
begin writeln('Calculate Smarandache function for all integers up to
',limit);
for count:=1 to limit do Smaran[count]:=0;
Smaran[limit+1]:=limit+1;
t:=time;
     {powers of 2}
s:=2;
i:=1;
is:=1;
pp:=1;
exit:=false;
repeat
repeat
pp:=pp*2;
Smaran[pp]:=s;
i:=i+1;
if 2*pp>limit then exit:=true;
until (i>is) or exit;
s:=s+2;
is:=invSpp(2,s);
until exit;
     {general case}
prime:=3;
repeat
tabsmarpp(prime,limit);
mcount:=1;
repeat
pwcount:=1;
multiple:=mcount*prime;
repeat
if multiple<=limit then
     if Smaran[multiple]=0 then
          Smaran[multiple]:=max(Smaran[mcount],Smarpp[pwcount]);
pwcount:=pwcount+1;
multiple:=mcount*Prpwr[pwcount];
until multiple>limit;
repeat
mcount:=mcount+1;
until Smaran[mcount]<>0;
until mcount*prime>limit;
pwcount:=1;
repeat
Smaran[Prpwr[pwcount]]:=Smarpp[pwcount];
pwcount:=pwcount+1;
until Prpwr[pwcount]>limit;
repeat
prime:=prime+1;
until Smaran[prime]=0;
until prime*prime>limit;
```

```
     {multiple case}
repeat mcount:=1;
multiple:=prime;
repeat
if multiple<=limit then
     if Smaran[multiple]=0 then
          Smaran[multiple]:=max(Smaran[mcount],prime);
repeat
mcount:=mcount+1;
until Smaran[mcount]<>0;
multiple:=mcount*prime;
until multiple>limit;
Smaran[prime]:=prime;
repeat
prime:=prime+1;
until Smaran[prime]=0;
until prime>limit/2;
     {largest primes}
count:=1;
repeat
if Smaran[count]=0 then Smaran[count]:=count;
count:=count+1;
until count>limit;
writeln((time/t)/100,'seconds');
end; {tabsmarand}

begin
tabsmarand(limit);
repeat
writeln('Enter start and finish integers for display of results');
read(st,fin);
if (st>1) and (st<=limit) and (fin<=limit) then
     for count :=st to fin do writeln(count,Smaran[count]);
until fin=1;
end. {Testsmarand}
```

A BRIEF HISTORY OF THE "SMARANDACHE FUNCTION" (II)

by Dr. Constantin Dumitrescu

{ We apologize, but the following conjecture that:
<u>the equation $S(x) = S(x+1)$, where S is the Smarandache Function, has no solutions</u>,
was not completely solved.
Any idea about it is wellcome.

See the previous issue of the journal for the first part of this article }

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ADDENDA:
New References concerninig this function (got by the editorial board after January 1, 1994):

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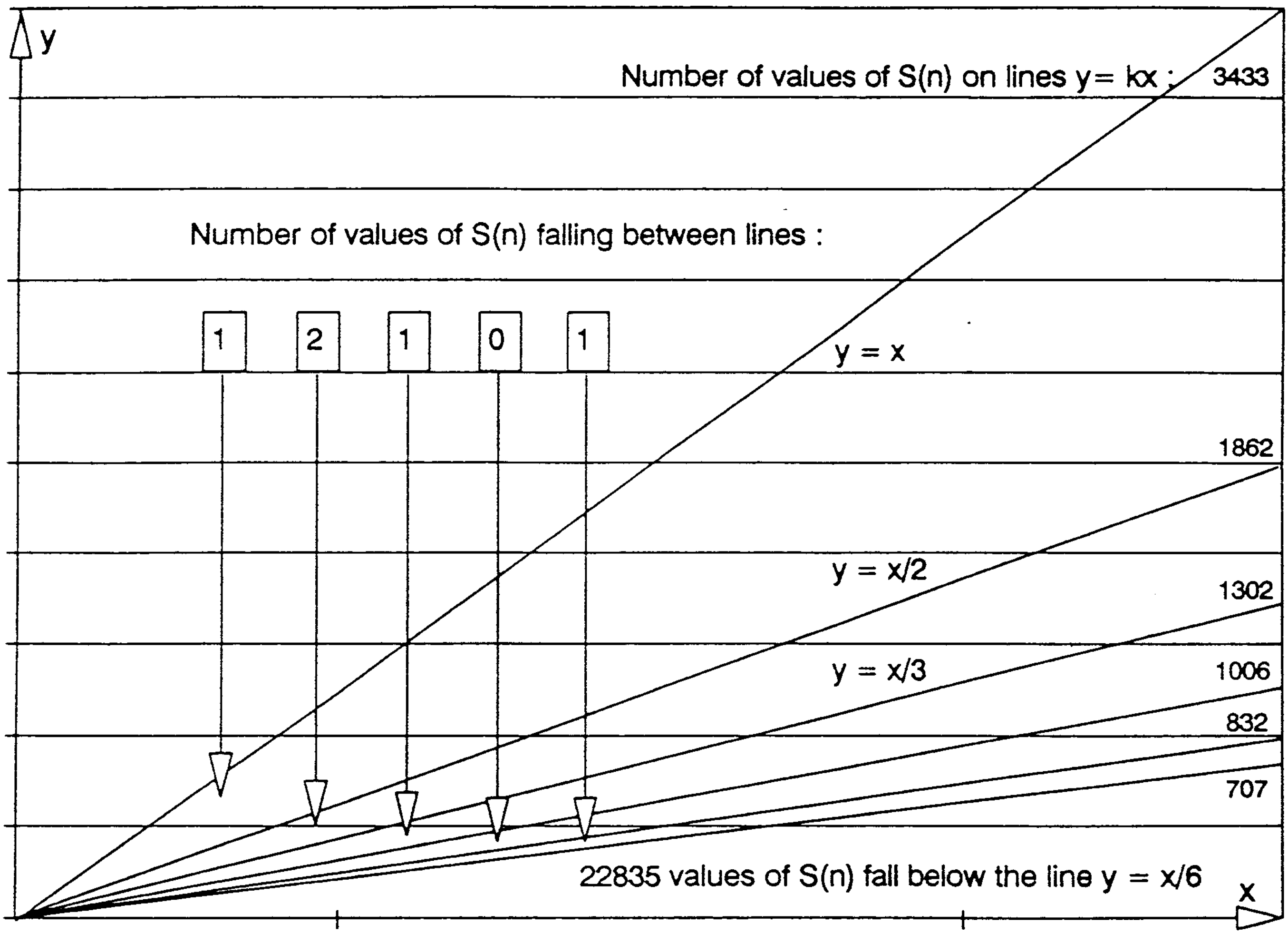
An Illustration of the Distribution of the Smarandache Function

by Henry Ibstedt

The cover illustration is a representation of the values of the Smarandache function for $n \leq 53$. The group at the back of the diagram essentially corresponds to $S(p)=p$, the middle group to $S(2p)=p$ $(p \neq 2)$ while the front group represents all the other values of $S(n)$ for $n \leq 53$.

Diagram 1. Distribution of S(n) up to n = 32000 (not to scale)



It may be interesting to take this graphical presentation a bit further. All the values of $S(n)$ for $n \leq 32000$ (conveniently chosen in order to use short integers only) have been sorted as shown in table 1. Of the 19114 points $(n,S(n))$ situated above the line $y = x/50$ only 61 points fall between lines. All of these of course correspond to cases where n is not square free. Diagram 1 illustrates this for the lines $y=x$, $y=x/2$, $y=x/3$, $y=x/4$, $y=x/5$ and $y=x/6$. The top line contains 3433 points $(n,S(n)$ although there are only 3432 primes less than 32000. This is because $(4,S(4))$ belongs to this line.

TABLE 1. On the distribution of the Smarandache Function S(n) for n≤32000.

N = number of values of S(n) on the line y=x/k, i.e. S(n)=n/k. The points (n,S(n)) are the only ones between lines y=x/k and y=x/(k+1) for k<50.

k	N	Points (n,S(n)) between lines:		
1	3433	(9, 6)		
2	1862	(16, 6)	(25, 10)	
3	1302	(49, 14)		
4	1006			
5	832	(121, 22)		
6	707	(169, 26)		
7	616	(45, 6)	(75, 10)	
8	550	(125, 15)	(289, 34)	
9	495	(361, 38)		
10	450	(147, 14)		
11	417	(529, 46)		
12	387			
13	359	(80, 6)		
14	336	(841, 58)		
15	321	(961, 62)		
16	301	(250, 15)	(343, 21)	(363, 22)
17	283	(175, 10)	(245, 14)	
18	273	(1369, 74)		
19	256	(507, 26)		
20	250	(243, 12)	(1681, 82)	
21	239	(1849, 86)		
22	227	(225, 10)		
23	213	(2209, 94)		
24	218			
25	204	(256, 10)	(867, 34)	
26	196	(2809,106)		
27	190	(605, 22)		
28	187	(1083, 38)		
29	176	(3481,118)		
30	179	(3721,122)		
31	163	(441, 14)	(625, 20)	
32	164	(686, 21)	(845, 26)	
33	159	(500, 15)	(4489,134)	
34	154	(1587, 46)		
35	154	(5041,142)		
36	153	(5329,146)		
37	139			
38	139	(539, 14)	(847, 22)	
39	136	(6241,158)		
40	139	(486, 12)	(1331, 33)	
41	125	(6889,166)		
42	133	(512, 12)	(1445, 34)	
43	119	(2523, 58)		
44	125	(7921,178)		
45	126	(637, 14)	(1183, 26)	
46	117	(2883, 62)		
47	109	(1805, 38)		
48	120	(729, 15)	(9409,194)	
49	114	(1089, 22)		
50	112			

Number of elements below y = x/50: 12774 .

PROBLEM (1)

by J. Rodriguez, Sonora, Mexico

Find a strictly increasing infinite series of integer numbers such that for any consecutive three of them the Smarandache Function is neither increasing nor decreasing.

*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

a) To solve the first part of this problem, we construct the following series:

p_3 , p_3+1, p_4 , p_4+1, ..., p_n , p_n+1, ...

where p_3 , p_4 , p_5 ... are the series of prime odd numbers 5, 7, 11 ...

Of course, $S(p_i) = p_i$ and $S(p_i+1) < p_i$, for any $i \geq 3$.

b) A way to look at this unsolved question is the following:

Because $S(p) = p$, for any prime number, we should get a large interval in between two prime numbers. A bigger chance is when p and q, the primes with that propriety, are very large (and $q \neq p + c$, where $c = 2, 4,$ or 6). In this case the series is finite. But this is not the optimum method!

The Smarandache Function is, generally speaking, increasing {we mean that for any positive integer k there is another integer $j > k$ such that $S(j) > S(k)$}. This property makes us to think that our series should be finite.

Calculating at random, for example, the series' width is at least seven, because:

for n = 43, 46, 57, 68, 70, 72, 120 then

S(n) = 43, 23, 19, 17, 10, 6, 5 respectively.

We are sure it's possible to find a larger series, but we worry if a maximum width does exist, and if this does: how much is it?

[Sorry, the author is not able to solve it!]

See: Mike Mudge, "The Smarandache Function" in the <Personal Computer World> journal, London, England, July 1992, page 420.

Solution of a problem by J. Rodriguez

by Pål Grønås

Problem: "Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing".

My intention is to prove that there exists series of arbitrary finite length with the properties described above.

To begin with, define $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and more generally, p_n = the nth prime. Now we have the following lemma:

Lemma: $p_k < p_{k+1} < 2p_k$ for all $k \in \mathbb{N}$. (Δ)

Proof: A theorem conjectured by Bertrand Russell and proven by Tchebychef states that for all natural numbers $n \geq 2$, there exists a prime p such that $n < p < 2n$. Using this theorem for $n = p_k$, we get $p_k < p < 2p_k$ $(\star)$ for at least one prime p. The smallest prime $> p_k$ is p_{k+1}, so $p \geq p_{k+1}$. But then it is obvious that $(\star)$ is satisfied by $p = p_{k+1}$. Hence $p_k < p_{k+1} < 2p_k$. □

This lemma plays an important role in the proof of the following theorem:

Theorem: Let n be a natural number ≥ 2 and define the series $\{x_k\}_{k=0}^{n-1}$ of length n by $x_k = 2^k p_{2n-k}$ for $k \in \{0, \ldots, n-1\}$. Then $x_k < x_{k+1}$ and $S(x_k) > S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. (Ω)

Proof: For $k \in \{0, \ldots, n-2\}$ we have the following equivalences: $x_k < x_{k+1} \Leftrightarrow 2^k p_{2n-k} < 2^{k+1} p_{2n-k-1} \Leftrightarrow p_{2n-k} < 2p_{2n-k-1}$ according to Lemma (Δ).

Futhermore $p_{2n-k} \geq p_{2n-(n-1)} = p_{n+1} \geq p_3 = 5 > 2$, so $(p_{2n-k}, 2) = 1$ for all $k \in \{0, \ldots, n-1\}$. Hence $S(x_k) = S(2^k p_{2n-k}) = \max\{S(2^k), S(p_{2n-k})\} = \max\{S(2^k), p_{2n-k}\}$. Consequently $p_{2n-k} \leq S(x_k) \leq \max\{2k, p_{2n-k}\}$ $(*)$ since $S(2^k) \leq 2k$.

Moreover we know that $p_{k+1} - p_k \geq 2$ for all $k \geq 2$ because both p_k and p_{k+1} are odd integers. This inequality gives us the following result:

$$\sum_{k=2}^{n-1} (p_{k+1} - p_k) = p_n - p_2 = p_n - 3 \geq \sum_{k=2}^{n-1} 2 = 2(n-2),$$

so $p_n \geq 2n - 1$ for all $n \geq 3$. In other words, $p_{n+1} \geq 2n + 1 > 2(n-1)$ for $n \geq 2$, i.e. $p_{2n-k} > 2k$ for $k = n-1$. The fact that p_{2n-k} increases and $2k$ decreases as k decreases from $n-1$ to 0 implies that $p_{2n-k} > 2k$ for all $k \in \{0, \ldots, n-1\}$. From this last inequality and $(*)$ it follows that $S(x_k) = p_{2n-k}$. This formula brings us to the conclusion: $S(x_k) = p_{2n-k} > p_{2n-k-1} = S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. □

Example: For $n = 10$ Theorem (Ω) generates the following series:

k	0	1	2	3	4	5	6	7	8	9
x_k	71	134	244	472	848	1504	2752	5248	9472	15872
$S(x_k)$	71	67	61	59	53	47	43	41	37	31

Problem (2)

J. Rodriguez, Sonora, Mexico

*Is it posible to extend the Smarandache Function from the integer numbers to the rational numbers (by finding then a *rational approach to the factorials, i.e. (3/2)! = ?*) ?

*More intriguering is to extend this function to the real numbers (by finding then a *real approach to the factorials, i.e.* $(\sqrt{5})! = ?$) ?

*Idem for the complex numbers (i.e. $(4 + 6i)! = ?$) ?

For example, we know that the Smarandache Function is defined as follows:

$$S : Z \setminus \{0\} \to N,\ S(n) \text{ is as the smallest integer such that } (S(n))! = 1x2x3x\ldots xS(n) \text{ is divisible by } n.$$

But what about S(1/2), or S(▯), or S(−i) are they equal to what ? It's interesting to try enlarging this funtion adopting in the same time new definitions for division and factorial, respectively.

Reference:

Mike Mudge, "The Smarandache Function", in the <Personal Computer World> journal, London, July 1992, p.420.

PROPOSED PROBLEM (3)

Let $\eta(n)$ be Smarandache Function: the smallest integer m such that m! is divisible by n. Calculate $\eta(p^{p+1})$, where p is an odd prime number.

Solution.

The answer is p^2, because:

$p^2! = 1\cdot 2\cdot \ldots \cdot p\cdot \ldots \cdot (2p)\cdot \ldots \cdot ((p-1)p)\cdot \ldots \cdot (pp)$, which is divisible by p^{p+1}.

Any another number less than p^2 will have the property that its factorial is divisible by p^k, with $k < p + 1$, but not divisible by p^{p+1}.

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PROPOSED PROBLEM (4)

Let m be a fixed positive integer. Calculate:

$$\lim_{i\to\infty} \eta(p_i^m)/p_i$$

where $\eta(n)$ is Smarandache Function defined as the smallest integer m such that m! is divisible by n, and p_i the prime series.

Solution:

We note by p_j a prime number greater than m. We show that

$\eta(p_i^m) = mp_i$, for any $i > j$:

if by absurd $\eta(p_i^m) = a < mp_i$ then

$a! = 1\cdot 2\cdot \ldots \cdot p_i \cdot \ldots \cdot (2p_i) \cdot \ldots \cdot ((m-k)p_i) \cdot \ldots \cdot a$, with $k > 0$, will be divisible by p_i^{m-k} but not by p_i^m.

Then this limit is equal to m.

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PROBLEM OF NUMBER THEORY (5)

by A. Stuparu, Vâlcea, Romania, and
D. W. Sharpe, Sheffield, England

Prove that the equation

$S(x) = p$, where p is a given prime number,

has just $D((p-1)!)$ solutions, all of them in between p and $p!$
[$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n, and $D(n)$ is the number of positive divisors of n].

PROOF (inspired by a remark of D. W. Sharpe) :

Of course the smallest solution is $x = p$, and the largest one is $x = p!$
Any other solution should be an integer number divided by p, but not by p^2 (because $S(kp^2) >= S(p^2) = 2p$, where k is a positive integer).

Therefore $x = pq$, where q is a a divisor of $(p-1)!$

Reference: "The Smarandache Function", by J. Rodriguez (Mexico) & T. Yau (USA), in <Mathematical Spectrum>, Sheffield, UK, 1993/4, Vol. 26, No. 3, pp. 84-5; Editor: D. W. Sharpe.

Examples (of D. W. Sharpe) :

$S(x) = 5$, then $x \in \{ 5, 10, 15, 20, 30, 40, 60, 120 \}$ (eight solutions).

$S(x) = 7$ has just 30 solutions, because $6! = 2^4 x 3^2 x 5^1$ and $6!$ has just $5x3x2 = 30$ positive divisors.

A PROBLEM CONCERNING THE FIBONACCI RECURRENCE (6)

by T. Yau, student, Pima Community College

Let S(n) be defined as the smallest integer such that (S(n))! is divisible by n (Smarandache Function). For what triplets this function verifies the Fibonacci relationship, i.e. find n such that

$$S(n) + S(n+1) = S(n+2) \ ?$$

Solution:
Checking the first 1200 numbers, I found just two triplets for which this function verifies the Fibonacci relationship:

$$S(9) + S(10) = S(11) \Leftrightarrow 6 + 5 = 11,$$

and

$$S(119) + S(120) = S(121) \Leftrightarrow 17 + 5 = 22.$$

How many other triplets with the same property do exist ?
(I can't find a theoretical proof ...)

Reference:

M. Mudge, "Mike Mudge pays a return visit to the Florentin Smarandache Function", in <Personal Computer World>, London, February 1993, p. 403.

GENERALISATION DU PROBLEME 1075* (7)

Soit n un nombre positif entier > 1.
Trouver Card{x, $\eta(x) = n$}. L'on a noté par $\eta(x)$ la Fonction Smarandache: qui est définie pour tout entier x comme le plus petit nombre m tel que m! est divisible par x.

M. Costewitz, Bordeaux, France

SOLUTION DU PROBLEME**:
(Ce problème est dans un sens une généralisation du problème 1075, publié dans l'*Elemente der Mathematik**.)
Soit $n = r_1^{d_1} \dots r_s^{d_s}$, la décomposition factorielle unique de ce nombre.

Calculons pour tout $1 \leq i \leq s$,

$$\sum_{j=1}^{\infty} [\, n/r_i^j \,] = e_i \geq d_i \geq 1,$$ où [a] signifie la partie entière de a.

C'est-à-dire: n! se divise par $r_i^{e_i}$, pour tout $1 \leq i \leq s$.
Nous nottons par M l'ensemble demande.
Biensur,

$$\bigcup_{i=1}^{s} \{\, r_i^{e_i}, r_i^{e_i-1}, \dots, r_i^{e_i-d_i+1} \,\} \subset M.$$

Nous nottons par R le membre gauche de l'inclusion antérieure, et par

$$R_i = \{\, r_i^{e_i}, r_i^{e_i-1}, \dots, r_i^{e_i-d_i+1} \,\},$$

$$R'_i = \{\, r_i^{e_i-d_i}, \dots, r_i, 1 \,\},$$ pour tous les i.

Soient $q_1, \dots, q_t$ tous les nombres premiers différents entre eux,

plus petits que n, et non-diviseurs de n. Il est clair que ceux-ci sont tous différents de $r_1, \dots, r_s$.

Construisons les suivantes suites finies:

$q_1, q_1^2, \dots, q_1^{f_1}$, tels que $\eta(q_1^{f_1}) < n < \eta(q_1^{f_1+1})$;
.
.
.
$q_t, q_t^2, \dots, q_t^{f_t}$, tels que $\eta(q_t^{f_t}) < n < \eta(q_t^{f_t+1})$;

$q_{t+1} > n;$
et

$$f_k = \sum_{j=1}^{\infty} [n/q_k^j], \text{ pour tous les } k.$$

Nous formons $q = \prod_{k=1}^{t} (1 + f_k)$ de combinaisons entre les nombres (éléments) de cettes suites, que nous réunissons dans un ensemble notté par Q.

Il est évident que chaque solution de l'équation $\eta(x) = n$ doit être de la forme: $a_i bc$, pour tous les i,

où $a_i \in R_i$, $b \in \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{s} R_j \right) \cup \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{s} R'_j \right)$, $c \in Q$.

Donc, le nombre des solutions pour l'équation demandée est égale à

$$q \sum_{i=1}^{s} d_i \prod_{\substack{j=1 \\ j \neq i}}^{s} \left(e_j + 1 \right).$$

[*Voir: Aufgabe 1075 par Thomas Martin, "**Elemente der Mathematik**", Vol. 48, No.3, 1993]

[**Solution complétée par les éditeurs (C. Dumitrescu)]

A PROBLEM OF MAXIMUM (8)

by T. Yau, student, Pima Community College

Let S(n) be defined as the smallest integer such that (S(n))! is divisible by n (Smarandache Function). Find:

$$\max\{ S(n)/n \},$$

over all composite integers $n \neq 4$.

Solution:

Let $n = p_1^{r_1} \cdots p_s^{r_s}$, its canonical factorial decomposition.

Because $S(n) = \max_{1 \le i \le s}\{ S(p_i^{r_i}) \} = S(p_j^{r_j}) \le p_j r_j$,

it's easy to see that n should have only a prime divisor for S(n)/n to become maximum. Therefore s = 1.

Then

$n = p^r$, where: p, r are integers, and p is prime.

$S(n)/n \le pr/p^r$. Hence p and r should be as small as possible, i.e.

p = 2 or 3 or 5, and r = 2 or 3.

By checking these combinations, we find

$n = 3^2 = 9$, whence $\max\{ S(n)/n \} = 2/3$

over all composite integers $n \neq 4$.

Reference:

M. Mudge, "Mike Mudge pays a return visit to the Florentin Smarandache Function", in <Personal Computer World>, London, February 1993, p. 403.

ALPHANUMERICS AND SOLUTIONS (9)

by T. Yau, student, Pima Community College

Prove that if N = 0 there are neither an operation " * " nor integers replacing the letters, for which the following statement:

```
  SMARANDACHE *
      FUNCTION *
            IN
 ----------------
= NUMBERTHEORY
```

is available.

Solution:

Of course " * " may not be an addition, because in that case "S" (as a digit) should be equal to "U", which involves N = 0. Contradiction.

[Same for a substraction.]

Nor a multiplication, because the product should have more that 12 digits.

Not a division, because the quotient should have less than 12 digits.

For other kind of operation, I think it's not necessary to check anymore.

Reference:

Mike Mudge, "The Smarandache Function" in the <Personal Computer World> journal, London, July 1992, p. 420.

THE MOST UNSOLVED PROBLEMS OF THE WORLD ON THE SAME SUBJECT

are related to the *Smarandache Function* in the Analytic Number Theory:

$S : Z^* \to N$, $S(n)$ is defined as the smallest integer such that $S(n)!$ is divisible by n.

The number of these unsolved problems concerning the function is equal to ... an infinity !! Therefore, they will never be all solved!

[See: Florentin Smarandache, "An Infinity of Unsolved Problems concerning a Function in the Number Theory", in the <Proceedings of the International Congress of Mathematicians>, Berkeley, California, USA, 1986]

TEACHING THE SMARANDACHE FUNCTION TO THE AMERICAN COMPETION STUDENTS

by T. Yau

The Smarandache Function is defined: for all non-null integers, n, to be the smallest integer such that (S(n))! is divisible by n [see 1, 2, 3].

In order to make students from the American competions to learn and understand better this notion, used in many east - european national mathematical competions, the author: calcutates it for some small numbers, establishes a few proprieties of it, and involves it in relations with other famous functions in the number theory.

It's important for the teachers to familiarize American students with the work done in other countries. (I would call it: multi - scientifical exchange.)

References:

1. Mike Mudge, "The Smarandache Function" in <Personal Computer World>, London, July 1992, p.420;
2. Debra Austin, "The Smarandache Function featured" in <Honeywell Pride>, Phoenix, Juin 22, 1993, p.8;
3. R. Muller, "Unsolved Problems related to Smarandache Function", Number Theory Publishing Co., Chicago, 1993.

						F					
1	11	6	4	6	11	1	2	4	5	12	6
2		13	10	3	4	2	7	1	5	7	
3	7	2	4	1	12	3	10	14	6	1	2
4			5	7	1	4	2	15	5		
5			16	5	14	5	12	3	4		
6				6	7	6	10	2			
7	17	9	6	4	6	7	16	6	11	18	2
8					7	8	10				
9				1	5	9	5	17			
10		19	4	3	13	2	17	3	4	5	
11				11	6	4	1	2			
12			6	10	15	2	20	4	6		
13				14	3	10	8	9			
14		13	6	11	1	3	4	5	6	10	
15					3	4	2				
						S					

Aritmogrif

Înlocuind numerele prin litere veți obține:

pe verticala F — S numele unei branșe matematice unde se încadrează *Funcția Smarandache* (2 cuv.), iar pe orizontale:

1)Proprietatea fundamentală ce contribuie la recunoașterea numerelor prime cu ajutorul acestei funcții (verb);

2)Prenumele autorului funcției;

3)"O infinitate de probleme ... referitoare la o funcție în Teoria Numerelor", articol stârnind interesul matematicienilor (vezi "*Proceedings of the International Congress of Mathematicians 1986*", Berkeley,CA);

4)Mulțime de numere pe care este definită funcția;

5)n pentru {S(n)}! în definiția funcției;

6)"... *Universității din Timișoara*", revista în care s'a publicat prima dată articolul "*O funcție în Teoria Numerelor*", 1980 (neart.);

7)Numele de familie al autorului funcției;

8)Număr întreg care nu aparține domeniului de definiție al funcției;

9)Județul în care se tipărește revista "*Analele Universității* din Timișoara";

10)Conf. dr. V. Seleacu, lect. dr. C. Dumitrescu, etc. care au format un grup de cercetare în cadrul Universității din Craiova privind proprietățile și aplicabilitatea acestei funcții (pl.);

11)"*Unsolved Problems related to the Smarandache Function*" de R. Muller, No. Th. Publ. Co., Chicago, 1993;

12)Ramura științifică incluzând Teoria numerelor;

13)Fascicul matematic;

14)De pildă {S(n)}!;

15)Aproximativ 3 asemenea unități de timp iau trebuit informaticianului suedez *Henry Ibstedt*, folosind un computer *dtk* având procesorul de 486/33 MHz în Turbo Basic Borland, pentru a calcula valorile *Funcției Smarandache* de la 1 până la ... 10^6!! In urma acestei "isprăvi" el a câștigat concursul organizat de omul de știință *Mike Mudge* din *Londra* asupra unor probleme deschise implicând *Funcția Smarandache* pentru revista "*Personal Computer World*" (Iulie, 2992, p. 420; Februarie, 1993, p. 403; August, 993, p. 495).

Autor: G. Dincu - *Drăgășani*, ROMANIA.

(Din revista "Abracadabra", Salinas, California, Noiembrie 1993, pp. 14-5, Editor : Ion Bledea)

A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc.

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